Kinematics & Dynamics of Linkages

Lecture 14: Velocity Analysis

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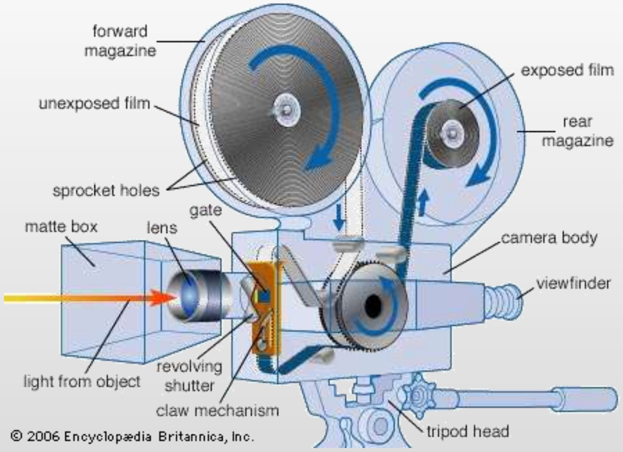
Introduction

- Velocity analysis involves determining "how fast" certain points on the links of a mechanism are traveling.
- Velocity is important because it associates the movement of a point on a mechanism with time. Often the timing in a machine is critical.

Example: The mechanism that "pulls" video film through a movie projector must advance the film at a rate of 30 frames per second



https://kids.britannica.com/students/assembly/view/90137

Linear Velocity

- Linear velocity (V) of a point is the linear displacement of that point per unit time.

- Linear Velocity is expressed as: $V = \lim_{\Delta t \to 0} \frac{dR}{dt}$

  and for short time periods: $V \cong \frac{\Delta R}{\Delta t}$

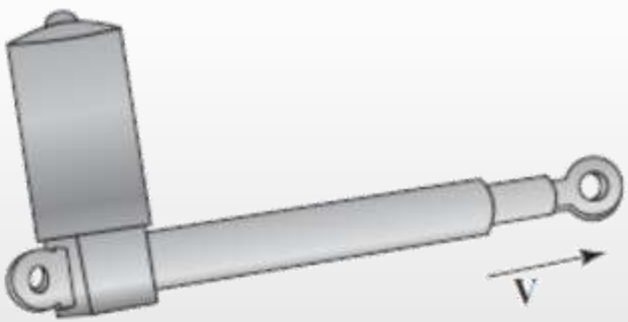
Example 6.1

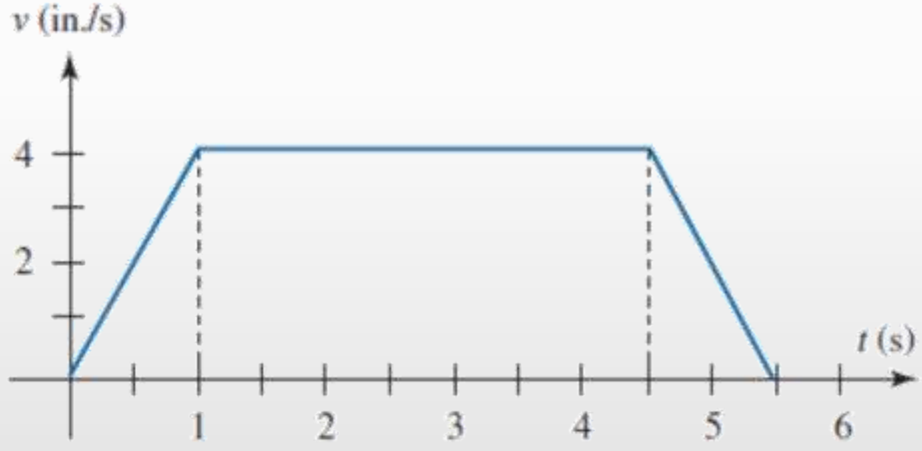
- Find the linear velocity of a crate on a conveyor that took 40 s to travers 25-ft to the left at a constant rate.

$$V_{crate} = \frac{\Delta R}{\Delta t} = \frac{25\ \text{ft}}{40\ \text{s}} = .625\ \text{ft/s}\left(\frac{12\ \text{in.}}{1\ \text{ft}}\right) = 7.5\ \text{in./s}$$

Example 6.2

- Determine the total displacement of the linear actuator that is programmed to extend according to the shown velocity profile.





Example 6.2 – Solution

- Displacement during the Speed-Up Portion of the Move

$$\Delta R_{\text{speed-up}} = \tfrac{1}{2}\left(V_{\text{steady-state}}\right)\left(\Delta t_{\text{speed-up}}\right) = \tfrac{1}{2}\left(4\text{ in./s}\right)[(1-0)\text{ s}] = 2\text{ in.}$$

- Displacement during the Steady-State Portion of the Move

$$\Delta R_{\text{steady-state}} = \left(V_{\text{steady-state}}\right)\left(\Delta t_{\text{steady-state}}\right) = \left(4\text{ in./s}\right)[(4.5-1)\text{ s}] = 14\text{ in.}$$

- Displacement during the Slow-Down Portion of the Move

$$\Delta R_{\text{slow-down}} = \tfrac{1}{2}\left(V_{\text{steady-state}}\right)\left(\Delta t_{\text{slow-down}}\right) = \tfrac{1}{2}\left(4\text{ in./s}\right)[(5.5-4.5)\text{ s}] = 2\text{ in.}$$

- Total Displacement during the Programmed Move

$$\Delta R_{\text{total}} = \Delta R_{\text{speed-up}} + \Delta R_{\text{steady-state}} + \Delta R_{\text{slow-down}} = 2 + 14 + 2 = 18\text{ in.}$$

Angular Velocity

- Angular velocity, ω, of a link is the angular displacement of that link per unit of time.

- Angular Velocity is expressed as: $\omega = \lim_{\Delta t \to o} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}$

  and for short time periods: $\omega \cong \frac{\Delta \theta}{\Delta t}$

Relation between angular and linear V

- For a link in pure rotation, the magnitude of the linear velocity of any point attached to the link is related to the angular velocity of the link. This relationship is expressed as

$$v = r\omega$$

where:

$v = |\mathbf{V}|$ = magnitude of the linear velocity of the point of consideration

r = distance from the center of rotation to the point of consideration

ω = angular velocity of the rotating link that contains the point of consideration

RPM conversion

- Conversion from RPM to rad/min

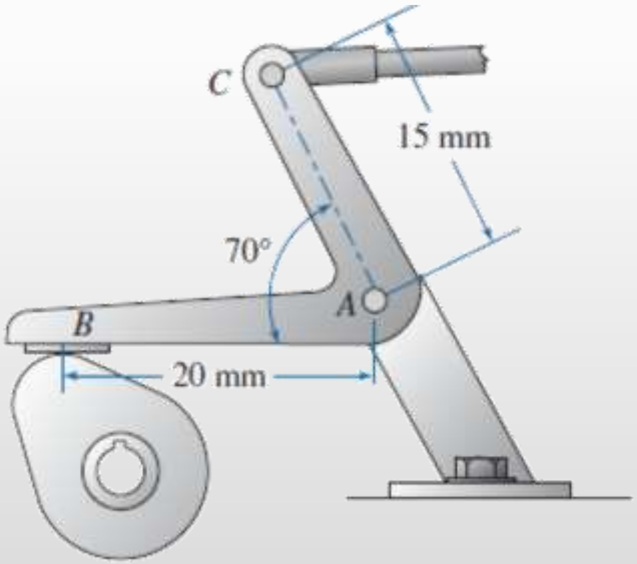
$$\omega(\text{rad/min}) = \left[\omega\,(\text{rad/min})\right]\left[\frac{2\pi\ \text{rad}}{\text{rev}}\right]$$
$$= 2\pi\left[\omega\,(\text{rad/min})\right]$$

- Conversion from RPM to rad/s

$$\omega\,(\text{rad/s}) = \left[\omega\,(\text{rev/min})\right]\left[\left(\frac{2\pi\ \text{rad}}{1\ \text{rev}}\right)\left(\frac{1\ \text{min}}{60\ \text{s}}\right)\right]$$
$$= \frac{\pi}{30}\left[\omega(\text{rev/min})\right]$$

Example 6.4

- Determine the angular velocity of the rocker plate at point **C** for the cam mechanism used to drive the exhaust port of an internal combustion engine. The cam forces point **B** upward at 30 mm/s.

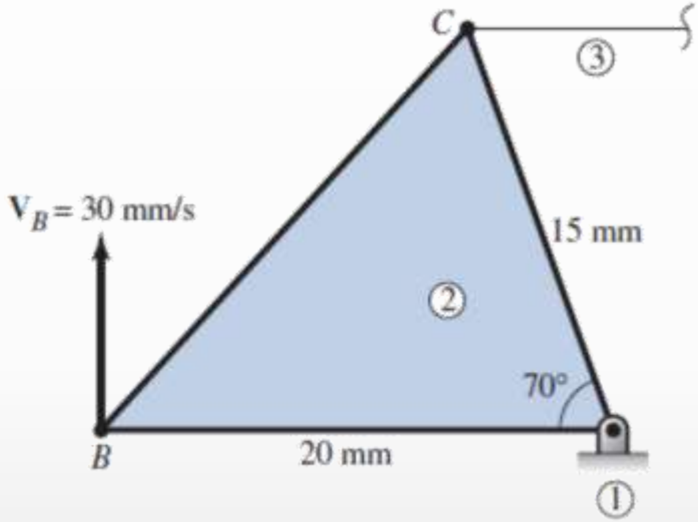


Example 6.4 – Solution

$$\omega_2 = \frac{v_B}{r_{AB}} = \frac{30 \text{ mm/s}}{20 \text{ mm}} = 1.5 \text{ rad/s}$$

$$v_C = r_{AC}\omega_2 = (15 \text{ mm})(1.5 \text{ rad/s}) = 22.5 \text{ mm/s}$$

$$\mathbf{V}_C = 22.5 \text{ mm/s} \angle 20^\circ$$



Relative Velocity

- Relative motion can be written mathematically as

$$\mathbf{V}_{B/A} = \mathbf{V}_B -> \mathbf{V}_A$$

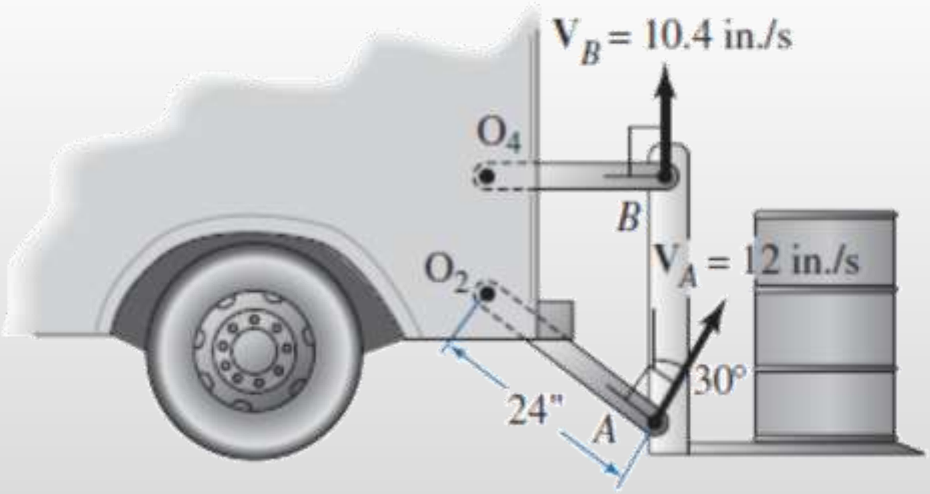
$\mathbf{V}_A$ = absolute velocity of point A

$\mathbf{V}_B$ = absolute velocity of point B

$\mathbf{V}_{B/A}$ = relative velocity of point B with respect to A
= velocity of point B "as observed" from point A

Example 6.5

- Point **A** on the cargo lift mechanism for a delivery truck has a velocity of 12 in./s, and point **B** has a velocity of 10.4 in./s. Determine the angular velocity of the lower link and the relative velocity of point **B** relative to point **A**.

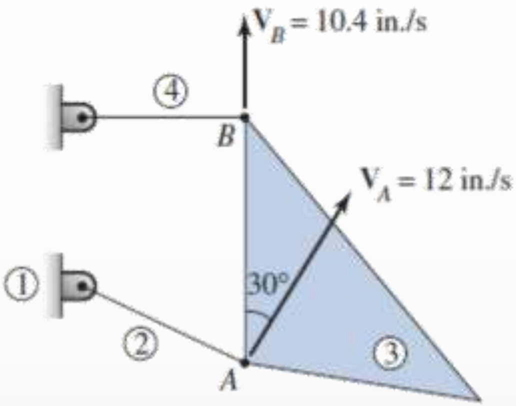


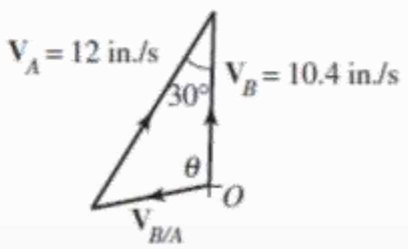
Example 6.5 – Solution

$$\omega_2 = \frac{v_A}{r_{AO_2}} = \frac{(12\text{ in./s})}{(24\text{ in.})} = 0.5\text{ rad/s}$$

$$\omega_2(\text{rev/min}) = \frac{30}{\pi}\left[\omega_2(\text{rad/s})\right] =$$

$$= \frac{30}{\pi}[0.5\text{ rad/s}] = 4.8\text{ rpm}$$





$$v_{B/A} = \sqrt{[v_A{}^2 + v_B{}^2 - 2(v_A)(v_B)(\cos 30°)]}$$

$$= \sqrt{(12\text{ in./s})^2 + (10.4\text{ in./s})^2 - 2(12\text{ in./s})(10.4\text{ in./s})(\cos 30°)} = 6.0\text{ in./s}$$

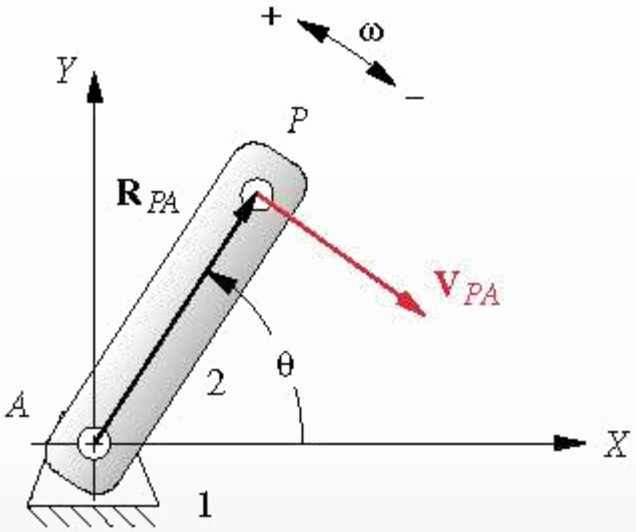
$$\theta = \sin^{-1}\left[\frac{(12\text{ in./s})}{6\text{ in./s}}\sin 30°\right] = 90°$$

$$V_{B/A} = 6.0\text{ in./s}$$

Vector representation

- Position of point P (with respect to A)

$$R_{PA} = pe^{j\theta}$$



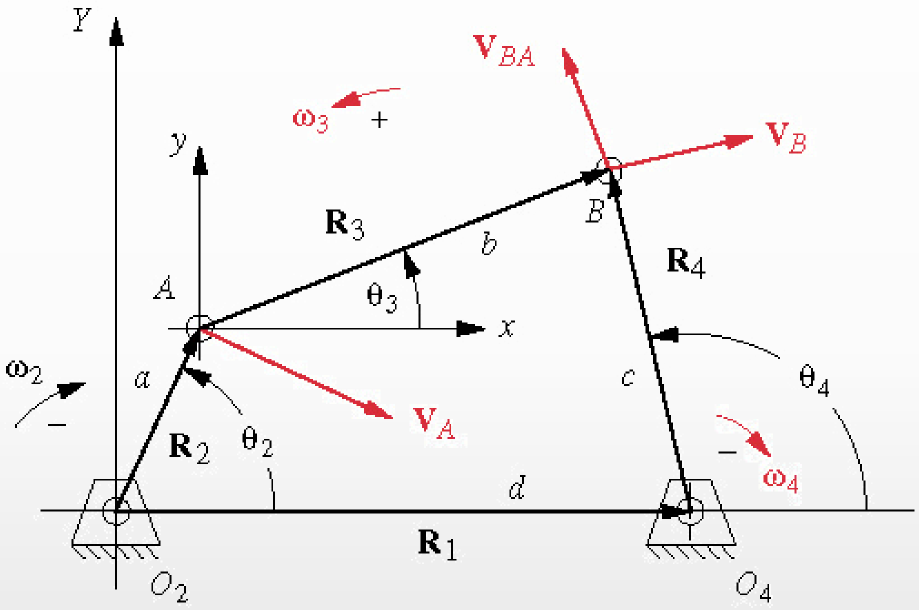
- Velocity of point P (with respect to A)

$$V_{PA} = \frac{dR_{PA}}{dt} = pje^{j\theta}\frac{d\theta}{dt} = p\omega je^{j\theta} = p\omega e^{j(\theta+90^0)}$$

- ω is + ccw and – cw

Analytical Solutions: 4Bar Mechanism

- Find the velocity of any point on the linkage



- The vector loop equation:

$$R_2 + R_3 - R_4 - R_1 = 0$$

$$ae^{j\theta_2} + be^{j\theta_3} - ce^{j\theta_4} - de^{j\theta_1} = 0 \quad \theta_1 = 0$$

$$ae^{j\theta_2} + be^{j\theta_3} - ce^{j\theta_4} - d = 0$$

Analytical Solutions: 4Bar Mechanism

- Differentiate the vector loop equation w/r to time and rearrange

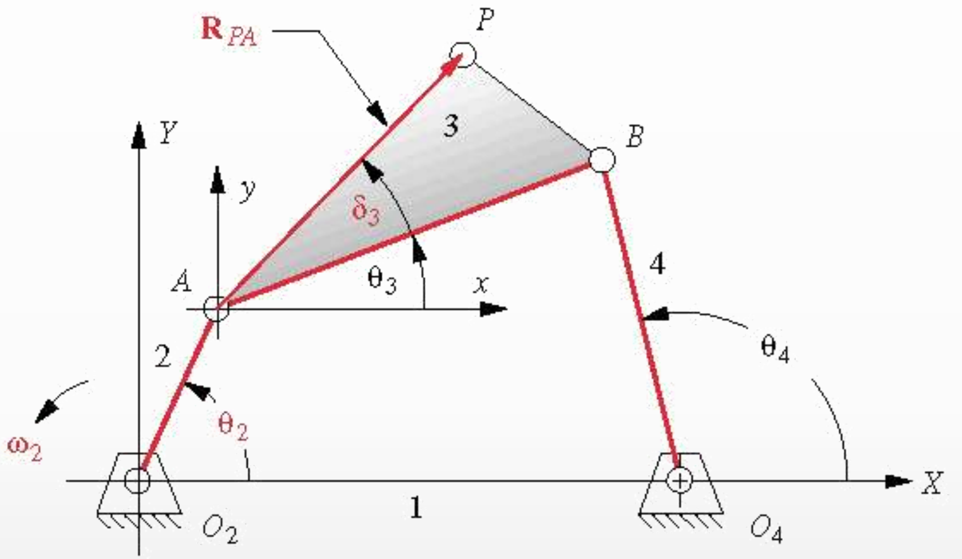
$$ja\omega_2 e^{j\theta_2} + jb\omega_3 e^{j\theta_3} - jc\omega_4 e^{j\theta_4} = 0$$

- Real part: $-a\omega_2 \sin\theta_2 - b\omega_3 \sin\theta_3 + c\omega_4 \sin\theta_4 = 0$
- Imaginary part: $a\omega_2 \cos\theta_2 + b\omega_3 \cos\theta_3 - c\omega_4 \cos\theta_4 = 0$

- Solution: $\omega_3 = \dfrac{a\omega_2}{b}\dfrac{\sin(\theta_4 - \theta_2)}{\sin(\theta_3 - \theta_4)}$ $\omega_4 = \dfrac{a\omega_2}{c}\dfrac{\sin(\theta_2 - \theta_3)}{\sin(\theta_4 - \theta_3)}$

Analytical Solutions: 4Bar Mechanism

Velocity of a point P on link 3



$$R_P = R_A + R_{PA} \qquad V_P = V_A + V_{PA}$$

$$R_{PA} = pe^{j(\theta_3+\delta_3)} \qquad R_A = ae^{j\theta_2}$$

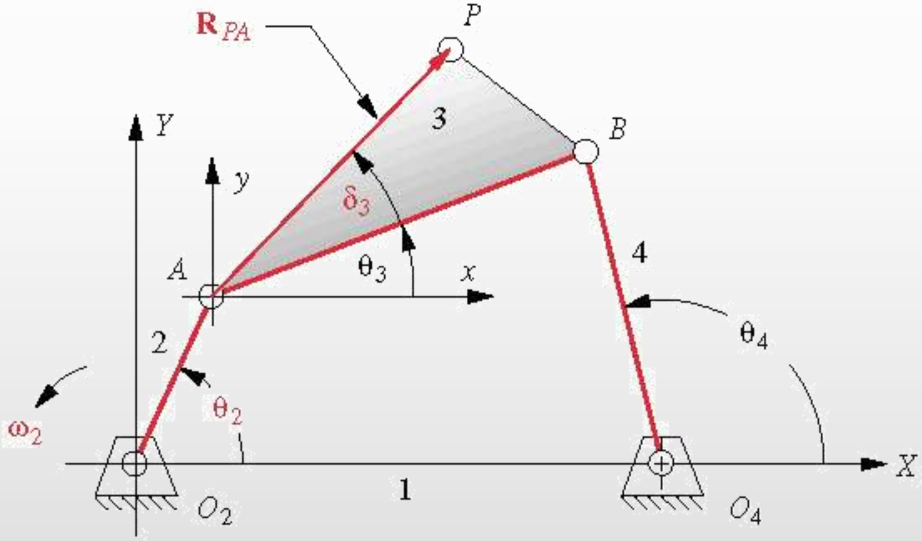
$$V_A = \frac{dR_A}{dt} = ja\omega_2 e^{j\theta_2} = a\omega_2\left(-\sin\theta_2 + j\cos\theta_2\right)$$

$$V_{PA} = \frac{dR_{PA}}{dt} = jp\omega_3 e^{j(\theta_3+\delta_3)} = p\omega_3\left(-\sin(\theta_3+\delta_3) + j\cos(\theta_3+\delta_3)\right)$$

$$V_P = -\left(a\omega_2\sin\theta_2 + p\omega_3\sin(\theta_3+\delta_3)\right) + j\left(a\omega_2\cos\theta_2 + p\omega_3\cos(\theta_3+\delta_3)\right)$$

4Bar Example

- Link 1 = 8″, Link 3 = 8″, Link 2 = 5″, Link 4 = 6″
- $\theta_2 = 75^o$, $\omega_2 = -50$ rad/s, $\delta+ = 300^o$, $R_{PA} = 9''$



- Determine: Velocity of Point P

4Bar Example – Solution

- **Step1**: Position analysis > find θ_3 and θ_4
- **Step2**: Velocity analysis > find ω_3 and ω_4
- **Step3**: Velocity of Point P

4Bar Example – Solution Step 1

$k_1 = (d/a) = 1.6$

$k_2 = (d/c) = 1.333$

$k_3 = (a^2 - b^2 + c^2 + d^2)/2ac = 1.017$

$k_4 = (d/b) = 1.0$

$k_5 = (c^2 - d^2 - a^2 - b^2)/2ab = -1.463$

$A = \cos\theta_2 - k_1 - k_2\cos\theta_2 + k_3 = -0.669$

$B = -2\sin\theta_2 = -1.932$

$C = k_1 - (k_2 + 1)\cos\theta_2 + k_3 = 2.013$

$D = \cos\theta_2 - k_1 + k_4\cos\theta_2 + k_5 = -2.545$

$E = -2\sin\theta_2 = -1.932$

$F = k_1 + (k_4 - 1)\cos\theta_2 + k_5 = 0.137$

a = length of link 2

b = length of link 3

c = length of link 4

d = length of link 1

$$\theta_3 = 2\tan^{-1}\left(\frac{-E \pm \sqrt{(E^2 - 4DF)}}{2D}\right)$$

= 7.5° or –79.0°

$$\theta_4 = 2\tan^{-1}\left(\frac{-B \pm \sqrt{(B^2 - 4AC)}}{2A}\right)$$

= 78.2° or –149.7°

4Bar Example – Solution Step 2

- Determine ω_3 & ω_4 use:

$$\omega_3 = \frac{a\omega_2}{b}\frac{\sin(\theta_4 - \theta_2)}{\sin(\theta_3 - \theta_4)}$$

$$\omega_4 = \frac{a\omega_2}{c}\frac{\sin(\theta_2 - \theta_3)}{\sin(\theta_4 - \theta_3)}$$

- Open configuration: $\theta_2 = 75$, $\theta_3 = 7.5$, $\theta_4 = 78.2$

- Solution: $\omega_3 = 1.85$ rad/s, $\omega_4 = -40.8$ rad/s

4Bar Example – Solution Step 3

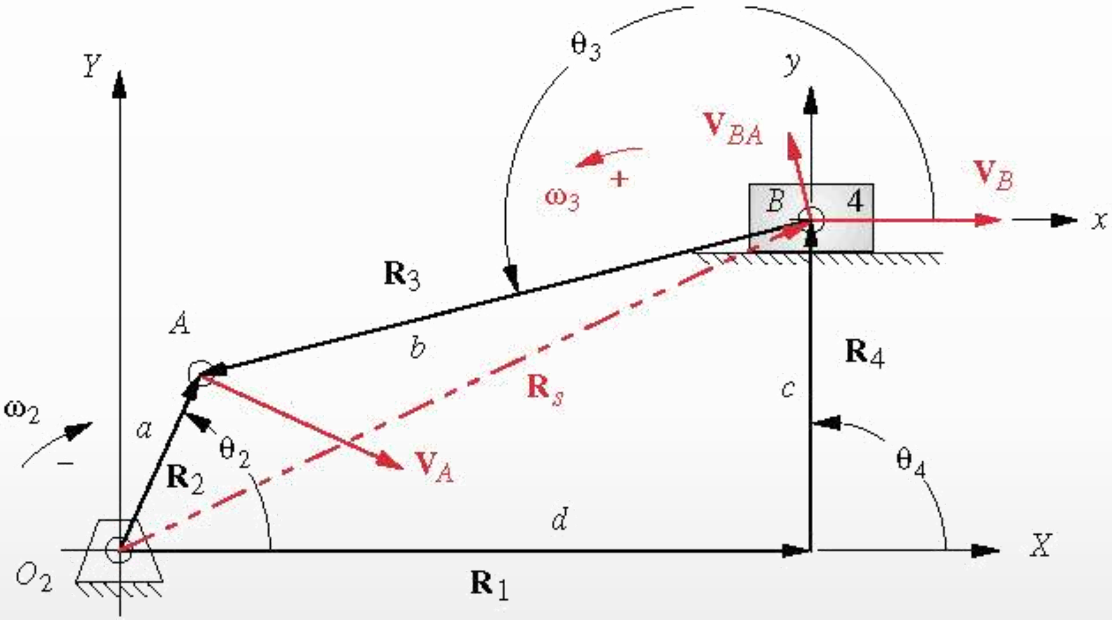
- Determine V_p & θ_4 using

$$V_P = V_A + V_{PA}$$
$$V_P = -\left(a\omega_2 \sin\theta_2 + p\omega_3 \sin(\theta_3 + \delta_3)\right) + j\left(a\omega_2 \cos\theta_2 + p\omega_3 \cos(\theta_3 + \delta_3)\right)$$

- Solution: $V_p = 254.7 - j\,54.61$ or 260.5 in/sec @ -12.1°

Analytical Solutions: Slider Crank

- Given ω_2
- Find ω_3 and V_B



- The vector loop equation:

$$R_2 - R_3 - R_4 - R_1 = 0$$

$$ae^{j\theta_2} - be^{j\theta_3} - ce^{j\theta_4} - de^{j\theta_1} = 0 \quad \theta_1 = 0, \quad \theta_4 = 90^0$$

$$ae^{j\theta_2} - be^{j\theta_3} - ce^{j90^0} - d = 0$$

Analytical Solutions: Slider Crank

- Differentiate the vector loop equation w/r to time and rearrange

$$j\omega_2 a e^{j\theta_2} - j\omega_3 b e^{j\theta_3} - \dot{d} = 0 \qquad \dot{d} = V_B$$

- Real part: $-a\omega_2 \sin\theta_2 + b\omega_3 \sin\theta_3 - \dot{d} = 0$
- Imaginary part: $a\omega_2 \cos\theta_2 - b\omega_3 \cos\theta_3 = 0$

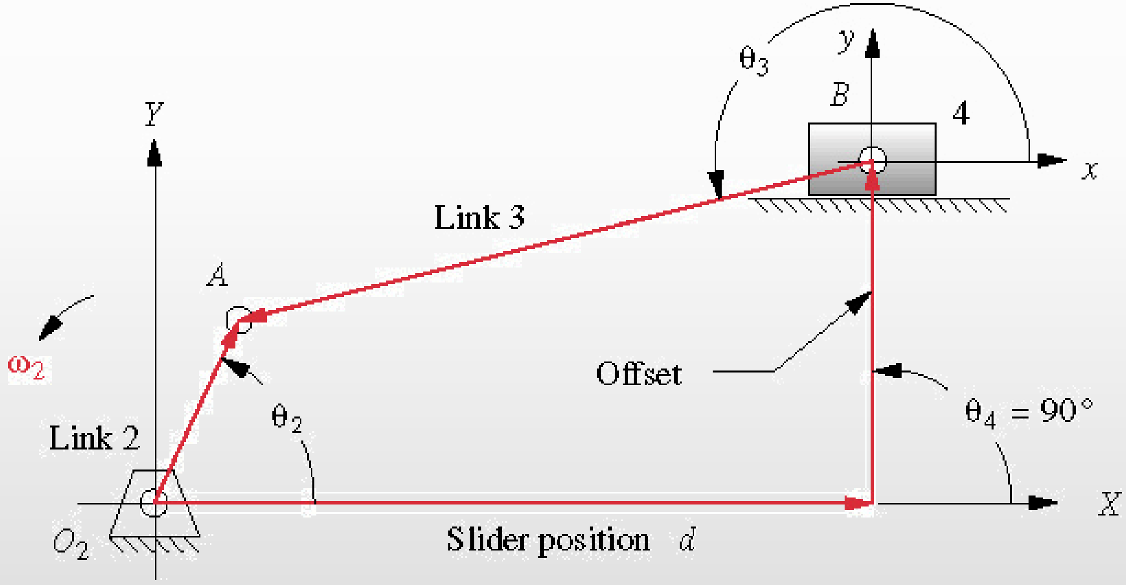
- Solution: $\omega_3 = \dfrac{a\omega_2}{b}\dfrac{\cos\theta_2}{\cos\theta_3}$

$$V_B = \dot{d} = -a\omega_2 \sin\theta_2 + b\omega_3 \sin\theta_3$$

Slider crank Example

Link 2 = 3", Link 3 = 8", Offset = 2"

θ_2= −30º, ω_2= −15 rad/s



Find Velocity V_A and V_B

Slider crank Example – Steps

- **Step1**: Position analysis > find θ_3
- **Step2**: Velocity analysis > find ω_3 and $\dot{d}$
- **Step3**: Calculate V_A and V_B

Slider crank Example – Solution Step 1

- Determine θ_3 from previous position analysis

a *= length of link 2 = 3 "*
b *= length of link 3 = 8 "*
c *= length of Offset = 2 "*

- Open Configuration

$$\theta_{3_1} = \sin^{-1}\left(\frac{a\sin\theta_2 - c}{b}\right) = \sin^{-1}\left(\frac{3\sin(-30) - 2}{8}\right) = 205.9^0$$

Slider crank Example – Solution Step 2

- Determine ω_3 and V_B

$$\omega_3 = \frac{a\omega_2}{b}\frac{\cos\theta_2}{\cos\theta_3}$$

$$V_B = \dot{d} = -a\omega_2\sin\theta_2 + b\omega_3\sin\theta_3$$

- Open Configuration: $\theta_2 = -30^o$, $\omega_2 = -15$ rad/s, $\theta_3 = 205.9^o$

- Solution: $\omega_3 = 5.42$ rad/s, $V_B = -41.44$ in/s

Slider crank Example – Solution Step 3

- Determine V_A

$$R_A = ae^{j\theta_2}$$

$$V_A = aje^{j\theta_2}\frac{d\theta_2}{dt} = a\omega_2 je^{j\theta_2}$$

$$V_A = a\omega_2 j(\cos\theta_2 + j\sin\theta_2) = -a\omega_2\sin\theta_2 + ja\omega_2\cos\theta_2$$

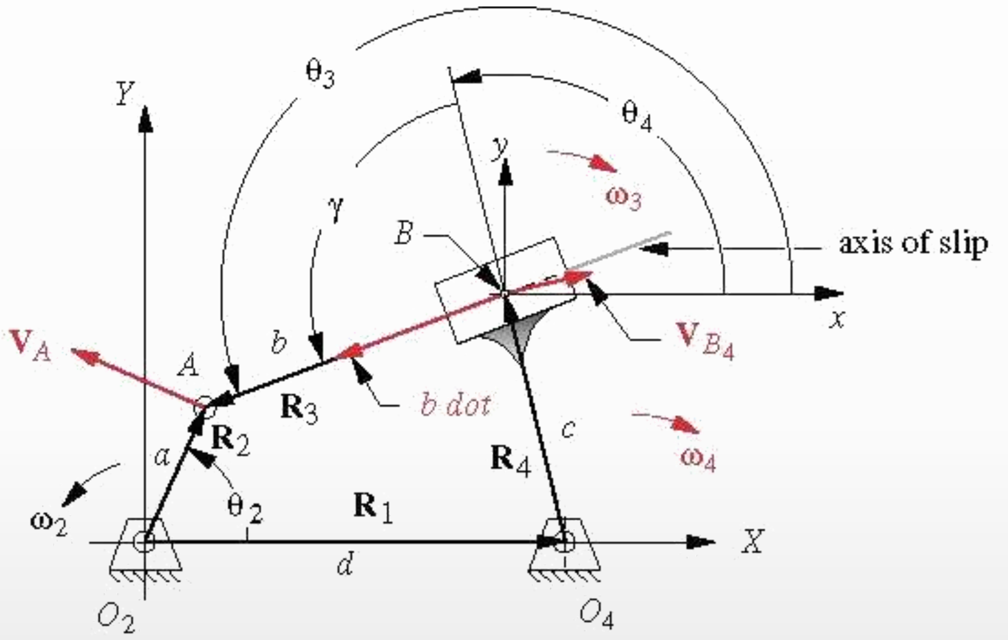
$$V_A = -3(-15)\sin(-30^o) + j\,3(-15)\cos(-30^o) = -22.5 - j\,38.97$$

- Open Configuration: $\theta_2 = -30^o$, $\omega_2 = -15$ rad/s, $\theta_3 = 205.9^o$

- Solution: $V_A = 45$ in/s @ 240^o

Analytical Solutions: Inverted Slider Crank

- Given ω_2
- Find ω3, ω4 and $\dot{b}$



- The vector loop equation:

$$R_2 - R_3 - R_4 - R_1 = 0$$

$$ae^{j\theta_2} - be^{j\theta_3} - ce^{j\theta_4} - de^{j\theta_1} = 0 \quad \theta_1 = 0^0$$

$$ae^{j\theta_2} - be^{j\theta_3} - ce^{j\theta_4} - d = 0$$

Analytical Solutions: Inverted Slider Crank

- Differentiate the vector loop equation w/r to time and rearrange

$$jae^{j\theta_2}\omega_2 - jbe^{j\theta_3}\omega_3 - \dot{b}e^{j\theta_3} - jce^{j\theta_4}\omega_4 = 0$$

- 3 Unknowns: $\omega_3, \quad \omega_4 \quad and \quad \dot{b}$
- But: $\theta_3 = \theta_4 \pm \gamma \quad differenti\,ate \quad \omega_3 = \omega_4$

- Substitute:

$$jae^{j\theta_2}\omega_2 - jbe^{j\theta_3}\omega_4 - \dot{b}e^{j\theta_3} - jce^{j\theta_4}\omega_4 = 0$$

Analytical Solutions: Inverted Slider Crank

- Differentiate the vector loop equation w/r to time and rearrange

Rotation A around B; Slip of B

$$\underbrace{jae^{j\theta_2}\omega_2}_{V_A} - \underbrace{\overbrace{jbe^{j\theta_3}\omega_3}^{\text{Rotation A around B}} \overbrace{-\dot{b}e^{j\theta_3}}^{\text{Slip of B}}}_{V_{AB}} - \underbrace{jce^{j\theta_4}\omega_4}_{V_B} = 0$$

V_A V_{AB} V_B

- Real part: $-a\omega_2 \sin\theta_2 + b\omega_4 \sin\theta_3 - \dot{b}\cos\theta_3 + c\omega_4 \sin\theta_4 = 0$
- Imaginary part: $a\omega_2 \cos\theta_2 - b\omega_4 \cos\theta_3 - \dot{b}\sin\theta_3 - c\omega_4 \cos\theta_4 = 0$

- Solution: $\omega_4 = \dfrac{a\omega_2 \cos(\theta_2 - \theta_3)}{b + c\cos\gamma}$ $\quad \dot{b} = \dfrac{-a\omega_2 \sin\theta_2 + \omega_4(b\sin\theta_3 + c\sin\theta_4)}{\cos\theta_3}$

Analytical Solutions: Inverted Slider Crank

- Slip Velocity: $V_{slip} = \dot{b} = V_{B34}$
- Point A Velocity: $V_A = ja\omega_2 e^{j\theta_2} = a\omega_2(-\sin\theta_2 + j\cos\theta_2)$
- Velocity of point B of link4 (Transmission velocity)

$$V_{B_4} = jc\omega_4 e^{j\theta_4} = c\omega_4(-\sin\theta_4 + j\cos\theta_4)$$

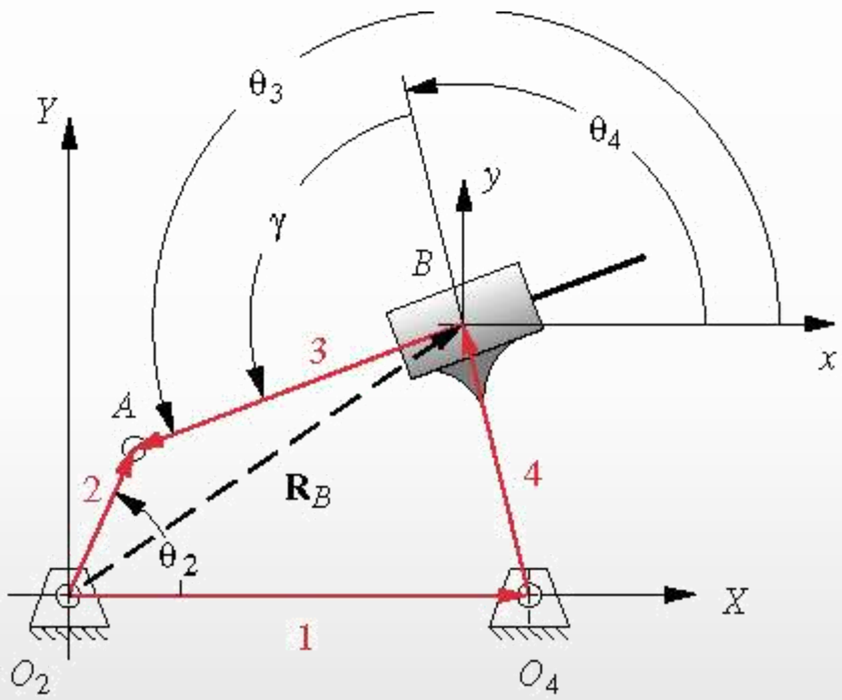
- Velocity of point B of link3 (Absolute Velocity)

$$V_{B_3} = V_{B_4} + V_{B_{34}} = V_{B_4} + V_{slip}(= V_{B,ref} + V_{B,slip})$$

Inverted slider crank Example

Link 1 = 3", Link 2 = 10", Link 4 = 6"

$\gamma = 45^o$, $\theta_2 = 45^o$, $\omega_2 = 24$ rad/s



ind V_A, V_B, and V_{Slip}

Inverted slider crank Example – Steps

- **Step1**: Position analysis > find θ_3 ,θ_4 and b
- **Step2**: Velocity analysis > find ω_3, ω_4, and $\dot{d}$
- **Step3**: Calculate $\dot{b}$

Inverted slider crank – Solution Step 1

- Determine θ_3, θ_4, and b from previous position analysis

$P = a \sin\theta_2 \sin\gamma + (a \cos\theta_2 - d)\cos\gamma = 7.879$
$Q = - a \sin\theta_2 \cos\gamma + (a \cos\theta_2 - d)\sin\gamma = - 2.121$
$R = -c \sin\gamma = - 4.243$
$S = R - Q = - 2.122$
$T = 2P = 15.758$
$U = Q + R = - 6.364$

a = length of link 2 = 10 ”
c = length of link 4 = 6 ”
d = length of link 1 = 3 ”

- Open Configuration

$\theta_4 = 2\arctan((- T + (T^2 - 4SU)^{.5})/2S) = 46.4°$
$\theta_3 = \theta_4 + \gamma = 46.4° + 45° = 91.4°$
$b = (a\sin\theta_2 - c\sin\theta_4)/\sin\theta_3 = 2.73”$

Inverted slider crank – Solution Step 2

- Determine ω_3, ω_4, & $\dot{b}$ using formulas

$$\omega_3 = \omega_4 = \frac{a\omega_2 \cos(\theta_2 - \theta_3)}{b + c\cos\gamma}$$

$$\dot{b} = \frac{-a\omega_2 \sin\theta_2 + \omega_4(b\sin\theta_3 + c\sin\theta_4)}{\cos\theta_3}$$

a = length of link 2 = 10 ”
c = length of link 4 = 6 ”
d = length of link 1 = 3 ”

- Open Configuration

θ_2= 45°, θ_3= 91.4°, θ_4= 46.4°, ω_2= 24 rad/s, γ= 45°, b= 2.73”

- Solution:

$$\omega_3 = \omega_4 = 23.7\ rad/\sec \qquad and \qquad \dot{b} = 83.77\ in/\sec$$

Inverted slider crank – Solution Step 3

- Determine $V_A, \quad V_{B_4} \quad and \quad V_{slip}$

$$V_A = a\omega_2(-\sin\theta_2 + j\cos\theta_2) = -169.7 + j169.7$$

$$V_{B_4} = c\omega_4(-\sin\theta_4 + j\cos\theta_4) = -102.9 + j71$$

$$V_{slip} = \dot{b} = 83.77 \ @ \ (\theta_3 + 180^o)$$

- Solution:

V_A = 240.0 in/s @ 135.0°

V_{B_4} = 142.2 in/s @ 136.6°

V_{slip} = 83.77 in/s @ 271.4°

LAU

Complex numbers example

- Calculate $\vec{F} = \vec{G} + \vec{H} - \vec{K}$

  Where $\vec{G} = 2@\theta \quad \vec{H} = 3@\beta \quad \vec{K} = 1@\phi$

- Solution $\vec{F} = \begin{bmatrix} 2\cos\theta \\ 2\sin\theta \end{bmatrix} + \begin{bmatrix} 3\cos\beta \\ 3\sin\beta \end{bmatrix} - \begin{bmatrix} 1\cos\phi \\ 1\sin\phi \end{bmatrix} = \begin{bmatrix} 2\cos\theta + 3\cos\beta - 1\cos\phi \\ 2\sin\theta + 3\sin\beta - 1\sin\phi \end{bmatrix}$

- Drive $\dot{\vec{F}} = \begin{bmatrix} -2\dot{\theta}\sin\theta - 3\dot{\beta}\sin\beta + 1\dot{\phi}\sin\phi \\ 2\dot{\theta}\cos\theta + 3\dot{\beta}\cos\beta - 1\dot{\phi}\cos\phi \end{bmatrix}$

- Using complex number approach we can reach same results